A REMARK ON MUMFORD'S COMPACTNESS THEOREM†

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ABSTRACT

It is shown that a recent compactness theorem for Fuchsian groups, due to Mumford, remains valid for groups containing elliptic and parabolic elements.

A Fuchsian group Γ is a discrete subgroup of the real Möbius group $G = SL(2,\mathbb{R})/\{\pm I\}$. We will establish the following extension of a recent result by Mumford.

THEOREM 1. *The set of conjugacy classes* $[\Gamma]$ *of Fuchsian groups* Γ, *such that* mes (G/Γ) *is* $\leqq \mu < \infty$ *and the absolute value of the trace of every hyperbolic element* γ *of* Γ *is* $\geqq 2 + \varepsilon > 2$, *is compact.*

We recall that Fuchsian groups Γ with mes $(G/\Gamma) < \infty$ are finitely generated. Hence the space of conjugacy classes $[\Gamma]$ of all such groups has a natural topology: a (distinguished) neighborhood V of Γ is determined by a sequence $\{\gamma_1, \cdots, \gamma_r\}$ of generators of Γ and a neighborhood v of the identity in G; a conjugacy class $[\Gamma']$ belong to V if and only if there is an isomorphism χ of Γ onto a $\Gamma'' \in [\Gamma']$ such that $\chi(\gamma)$ is parabolic if and only if γ is, and $\chi(\gamma_j) \circ \gamma_j^{-1} \in v$ for $j = 1, \cdots, r$.

In [5] Mumford proved a general compactness theorem and obtained, as a corollary, a statement analogous to Theorem 1, under the additional hypotheses that all groups Γ considered are torsion free and all quotients G/Γ are compact. He stated that the corollary can be obtained by an elementary argument. Our proof of Theorem 1 is an extension of this argument.

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A Fuchsian group Γ acts on the upper half plane U as a group of conformal automorphisms; the condition $\text{mes}\,(G/\Gamma) \leqq \mu$ is equivalent to the condition $\iint_{U/\Gamma} y^{-2}\, dxdy \leqq c\mu$ where c is a universal constant. Every group Γ satisfying this condition has a *signature*

$$\sigma = (p, n; \nu_1, \cdots, \nu_n) \tag{1}$$

where p and n are integers, the ν_j are integers or the symbol ∞, and

$$p \geqq 0, \quad n \geqq 0, \quad 2 \leqq \nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_n \leqq \infty\,,$$

(2)

$$A(\sigma) = 2\pi(2p - 2 + n - \frac{1}{\nu_1} - \cdots \frac{1}{\nu_n}) > 0.$$

We have that $\iint_{U/\Gamma} y^{-2}\, dxdy = A(\sigma)$, the Riemann surface U/Γ is a compact surface of genus p with n_∞ points removed, n_∞ being the number of times ∞ occurs among the symbols $\nu_1, \cdots, \nu_n$, and Γ has precisely n non-conjugate in Γ maximal cyclic elliptic or parabolic subgroups, the order of these subgroups being $\nu_1, \cdots, \nu_n$. Note that U/Γ is compact if and only if G/Γ is, and if and only if $n = 0$ or $\nu_n < \infty$; we call such signatures of compact type.

A Fuchsian group with signature (1) is said to represent the *configuration*

$$\Sigma = (S; P_1, \cdots, P_n) \tag{3}$$

where S is a compact Riemann surface of genus p and $P_1, \cdots, P_n$ are distinct points on S, if there is a conformal bijection $f\colon U/\Gamma \to S - \{P_{n-n_\infty+1}, \cdots, P_n\}$ such that $f^{-1}(P_j)$ is the image under $U \to U/\Gamma$ of a point $z_j \in U$ fixed under a maximal cyclic subgroup of Γ of order ν_j, $j = 1, \cdots, n - n_\infty$. The group determines the configuration Σ except for a conformal equivalence and a permutation of the "ramification points" P_j in which each P_i is taken into P_k with $\nu_i = \nu_k$. Conversely, given σ and Σ, satisfying (1), (2) and (3), there is a Fuchsian group Γ of signature σ, determined up to conjugacy in G, which represents Σ. This is the *limit circle theorem* of Klein and Poincaré.

We denote by $X(\sigma)$ the set of conjugacy classes $[\Gamma]$ of Fuchsian groups Γ with signature σ. (If $\sigma = (p, 0)$ $p > 1$, then $X(\sigma)$ is the space of moduli of compact Riemann surfaces at genus p.) One verifies, for instance by using quasiconformal mappings, that the spaces $X(\sigma)$, with their natural topologies, are metrizable. More precisely, the topology of $X(\sigma)$ can be derived from the Teichmüller metric defined as follows: the distance between $[\Gamma]$ and $[\Gamma']$ is the smallest number α

such there is a quasiconformal automorphism ω of U, with dilatation e^{α} and with $\omega\Gamma\omega^{-1} = \Gamma'$. The spaces $X(\sigma)$ also have natural complex structures (see [2] and the references given there) which we need not use here.

Theorem 1 is equivalent to the following

THEOREM 1′. *The subset of* $X(\sigma)$ *corresponding to groups* Γ *such that* $|\text{trace}\,\gamma| \geqq 2 + \varepsilon > 2$ *for all hyperbolic* $\gamma \in \Gamma$, *is compact.*

We denote by diam Γ the diameter of U/Γ measured in the Riemannian metric induced on U/Γ by the Poincaré metric $|dz|/y$ in U. This diameter is finite if and only if U/Γ is compact.

LEMMA 1. *Let* σ *be of compact type. The subset of* $X(\sigma)$ *corresponding to groups* Γ *with* $\operatorname{diam}\Gamma \leqq \alpha < \infty$ *is compact.*

PROOF. Let Γ be a group of signature (1), and let $z_0 \in U$ be not a fixed point of an elliptic element of Γ. The *Dirichlet region* $\Pi(\Gamma, z_0)$ is the set of all $z \in U$ such that the Poincaré distance from z_0 to z is not greater than that from z_0 to $\gamma(z)$, $\gamma \in \Gamma$. We recall that $\Pi(\Gamma, z_0)$ is a fundamental (non-Euclidean) polygon for Γ, that the interior angles at all vertices of $\Pi(\Gamma, z_0)$ are $\leqq \pi$, with equality possible only if the vertex is fixed under an involution in Γ, and that the sides of $\Pi(\Gamma, z_0)$ are pairwise identified by elements $\gamma_1, \cdots, \gamma_q$ of Γ which generate the group.

The number $2q$ of sides is subject to *Fricke's inequality* (see [4], p. 262)

$$q \leqq 6p + 2n - 3. \tag{4}$$

(For the sake of completeness, we sketch the argument. Let m be the number of non-equivalent accidental vertices of $\Pi(\Gamma, z_0)$, that is, of vertices which are not elliptic fixed points. Then $2q \geqq 3m + n$. On the other hand, $2 - 2p = m + n - q + 1$, by the formula for the Euler characteristic. Inequality (4) follows.)

Let $\{\Gamma_j\}$ be a sequence of Fuchsian groups of signature σ, with $\operatorname{diam}\Gamma_j \leqq \alpha$. Lemma 1 will be proved if we find a subsequence of $\{[\Gamma_j]\}$ which converges in $X(\sigma)$.

Let z_0 be a point in U not fixed by an elliptic element of any Γ_j, and set $\Pi_j = \Pi(\Gamma_j, z_0)$. Then all Π_j lie in a closed non-Eulidean disc about z_0 of radius $\alpha/2$. Since all Π_j have non–Euclidean area $A(\sigma)$, it is easy to conclude that there is a $\beta > 0$ such that every Π_j contains a non-Euclidean disc of radius β, about some center ζ_j.

(Indeed, let x_j be a diameter of Π_j of maximal length a_j and y_j a segment of maximal length b_j orthogonal to x_j and contained in Π_j; all geometric terms refer to non-Euclidean geometry. Since $a_j \leqq \alpha$, the area of Π_j is $\leqq \phi(b_j)$, a continuous decreasing function, depending only on α, with $\phi(0) = 0$. Hence there is an α', $0 < \alpha' < \infty$, such that $b_j \geqq \alpha'$. Thus Π_j contains an equilateral right triangle whose legs have length $\alpha'/2$. Such a triangle contains a disc of some fixed radius β.)

Selecting if need be a subsequence, we may assume that $\zeta = \lim \zeta_j$ exists and that there is a number q and there are generators $\gamma_1, \cdots, \gamma_q$ of Γ_1 and isomorphisms $\chi_j \colon \Gamma_1 \to \Gamma_j$ with the following properties. The $\gamma_1, \cdots, \gamma_q$ identify the sides of Π_1. The $\chi_j(\gamma_1), \cdots, \chi_j(\gamma_q)$ identify the sides of Π_j. The limits $\hat{\gamma}_i = \lim \chi_j(\gamma_i)$ exist in G, $i = 1, \cdots, q$.

Let $\hat{\Gamma}$ be the group generated by $\hat{\gamma}_1, \cdots, \hat{\gamma}_q$ and let $\hat{\chi} \colon \Gamma_1 \to \hat{\Gamma}$ be the epimorphism which takes γ_i into $\hat{\gamma}_i$, $i = 1, \cdots, q$. Let γ be in Γ_1, and not the identity, and set $\hat{\gamma} = \hat{\chi}(\gamma)$. The non-Euclidean distance between ζ and $\hat{\gamma}(\zeta)$ is at least β. Using this remark one verifies that $\hat{\chi}$ is an isomorphism, that $\hat{\Gamma}$ is discrete, that $\hat{\Gamma}$ is a Fuchsian group of signature σ, and that $\lim [\Gamma_j] = [\hat{\Gamma}]$.

LEMMA 2. *Let* $\sigma = (p, n; v_1, \cdots, v_n)$ *and* $\hat{\sigma} = (p, n; \hat{v}_1, \cdots, \hat{v}_n)$ be *signatures such that* $v_j = v_k$ *if and only if* $\hat{v}_j = \hat{v}_k$. *Then there is a canonical topological bijection* $X(\sigma) \to X(\hat{\sigma})$.

PROOF. Map every $[\Gamma] \in X(\sigma)$ onto a $[\hat{\Gamma}] \in X(\hat{\sigma})$ such that Γ and $\hat{\Gamma}$ represent the same configuration. By the limit circle theorem, this is a well defined bijection. Every known proof of the limit circle theorem can be used to show continuity. It is particularly simple to use the proof by quasiconformal mappings (see [1]).

To every Fuchsian group Γ we associate a closed Γ invariant subset $\Delta(\Gamma)$ of U defined as follows. For every maximal parabolic subgroup $\Gamma_j \subset \Gamma$ with fixed point ζ_j, let $H(\Gamma_j)$ be the domain interior to a horocycle through ζ_j, chosen so that the quotient $H(\Gamma_j)/\Gamma_j$ has Poincaré area equal to 1. If Γ_j is generated by $z \mapsto z + 1$ (which can be achieved by conjugation), then $H(\Gamma_j)$ is the half-plane $y > 1$. It is known that two points in $H(\Gamma_j)$ are Γ equivalent only if they are Γ_j equivalent, and that $H(\Gamma_j) \cap H(\Gamma_k) = \varnothing$ for $\Gamma_j \neq \Gamma_k$.

We denote by $\Delta(\Gamma)$ the complement in U of the union of all $H(\Gamma_j)$.

(Since the properties of the $H(\Gamma_j)$ stated above are important for our argument, we recall the proof. It suffices to show that if Γ contains the element $\gamma_0(z) = z + 1$, it cannot contain an element $\gamma(z) = (az + b)/(cz + d)$ with $a, b, c, d, \in \mathbb{R}$, $ad - bc = 1$ and $0 < c < 1$. Assume it does, and set $\gamma_m = \gamma^m \circ \gamma_0 \circ \gamma^{-m}$. Then $\gamma_m(z)$

$= (a_m z + b_m)/(c_m z + d_m)$ with $a_m, b_m, c_m, d_m \in \mathbb{R}$, $a_m b_m - c_m d_m = 1$, and $|c_m| = c^m \to 0$ as $m \to \infty$. This contradicts the discreteness of Γ.)

If U/Γ has finite measure, $\Delta\Gamma/\Gamma$ is connected and compact. The reduced diameter, $\text{diam}^{\#}\Gamma$, is defined as the diameter of $\Delta(\Gamma)/\Gamma$ in the Riemannian metric induced by the Poincaré metric on U, that is, as the supremum of the infima of lengths of curves in $\Delta(\Gamma)/\Gamma$ joining pairs of distinct points.

If U/Γ is compact, then $\text{diam}^{\#}\Gamma = \text{diam}\,\Gamma$.

LEMMA 3. *Let σ and $\hat{\sigma}$ be as in Lemma 2. Assume in addition that $\hat{\sigma}$ is of compact type and that $\hat{\nu}_j \leqq \nu_j$, $j = 1, \cdots, n$.* Let Γ *and* $\hat{\Gamma}$ *be two Fuchsian groups with signatures σ and $\hat{\sigma}$, respectively, which represent the same configuration. Then*

$$\text{diam}\,\hat{\Gamma} \leqq \text{diam}^{\#}\Gamma + c \tag{5}$$

where the constant c depends only on σ and $\hat{\sigma}$.

PROOF. Let $\Sigma = (S; P_1, \cdots, P_n)$ be the configuration represented by Γ and by $\hat{\Gamma}$. The Poincaré metric in U induces, via the mappings

$$U \to U/\Gamma \to S - \{P_{n-n_\infty+1}, \cdots, P_n\}, \qquad U \to U/\hat{\Gamma} \to S, \tag{6}$$

real analytic Riemannian metrices $ds = \lambda(t)|dt|$ and $d\hat{s} = \hat{\lambda}(t)|dt|$ on $S - \{P_1 \cdots, P_n\}$; here t is a local parameter. Since the Poincaré metric has curvature (-1), we have that $\Delta \log \lambda = \lambda^2$, $\Delta \log \hat{\lambda} = \hat{\lambda}^2$. If t is a local parameter which vanishes at a point P_i, then for $t \to 0$,

$$\begin{aligned} \lambda(t) &\sim \text{const.}\,|t|^{-1+1/\nu} \quad \text{if} \quad \nu_j = \nu < \infty, \\ \lambda(t) &\sim |t|^{-1}(-\log|t|)^{-1} \quad \text{if } \nu_j = \infty\,. \end{aligned} \tag{7}$$

Similar formulas hold for $\hat{\lambda}$. The hypothesis $\hat{\nu}_j \leqq \nu_j$, $j = 1, \cdots, n$, together with standard arguments form the theory of quasi-linear elliptic partial differential equations of second order, imply that

$$d\hat{s} \leqq ds. \tag{8}$$

The set $\Delta(\Gamma)$ corresponds on S to a set D_0; the complement $S - D_0$ has n_∞ components $D_1, \cdots, D_{n_\infty}$. Denote the diameters of these sets, in the $d\hat{s}$ metric, by $\hat{\delta}_0, \hat{\delta}_1, \cdots, \hat{\delta}_{n_\infty}$. By the inequality just established, $\hat{\delta}_0 \leqq \text{diam}^{\#}\Gamma$; thus (5) will be proved once we obtain an estimate

$$\hat{\delta}_i \leqq c', \qquad i = 1, \cdots, n_\infty \tag{9}$$

with fixed c'. To get this estimate, note that each D_i is the image, under the second mapping (6), of a Jordan domain $\hat{D}_i$ in U whose boundary consists of two non-Euclidean segments of the same length r, forming an acute angle $2\pi/\hat{v}_n$ with vertex ξ_i, and of a smooth arc C_i joining the other two endpoints, ξ_i' and ξ_i'', of these segments. The second mapping (6) is one-to-one on $\hat{D}_i$ and on C_i, except for the endpoints of C_i. By (8), the non-Euclidean length of C_i is not greater than the length of the boundary curve of D_i in the ds metric, and the latter is easily seen to be 1. On the other hand, the non-Euclidean length of C_i is at least the non-Euclidean distance between ξ_i' and ξ_i''. This gives an upper bound, call it $\hat{r}$, for r, depending only on $\hat{v}_n$. Thus the non-Euclidean diameter of $\hat{D}_i$ is at most $1 + 2\hat{r}$, and (9) follows.

LEMMA 4. *The subset of* $X(\sigma)$ *corresponding to groups* Γ *with* $\text{diam}^{\#}\Gamma \leqq \alpha < 0$ *is compact.*

PROOF. Let $\hat{\sigma}$ be as in Lemma 3. The subset considered is closed. Its image in $X(\hat{\sigma})$ under the canonical mapping of Lemma 2 lies in a compact set, in view of Lemmas 3 and 1.

LEMMA 5. *Let* Γ *be a Fuchsian group of signature* σ, *and assume that for every hyperbolic* $\gamma \in \Gamma$ *we have that* $|\text{trace}\ \gamma| \geqq 2 + \varepsilon > 0$. *Then*

$$\text{diam}^{\#}\Gamma \leqq c/\varepsilon \tag{10}$$

where c depends only on σ.

For $\sigma = (p, 0)$, $p > 1$, this is Mumford's Corollary 1.

PROOF. We assume first that Γ has no torsion. Then the metric ds induced on U/Γ by the Poincaré metric in U is complete. Let $n = n_\infty$ be the number of non-conjugate maximal parabolic subgroups of Γ; this is also the number of components $D_1, \cdots, D_n$ of $U/\Gamma - \Delta(\Gamma)/\Gamma$.

We set

$$\varepsilon_1 = 2\,\text{arccosh}\,(1 + \frac{\varepsilon}{2}), \tag{11}$$

so that $\varepsilon_1 \sim 2\sqrt{\varepsilon}$ for $\varepsilon \to 0$. Without loss of generality we assume that

$$0 < \varepsilon_1 < 1. \tag{12}$$

The condition $|\text{trace}\ \gamma| \geqq 2 + \varepsilon$ for all hyperbolic γ in Γ means that for every such γ, and every $z \in U$, the non-Euclidean distance between z and $\gamma(z)$ is at least ε_1. In view of the definition of $\Delta(\Gamma)$ and condition (12), the same is true for $z \in \Delta(\Gamma)$

and *every* γ in Γ distinct from the identity. This means, as one sees at once, that every rectifiable closed curve in U/Γ which intersects $\Delta(\Gamma)/\Gamma$ and has non-Euclidean length $< \varepsilon_1$ is homotopic to a point. It follows that every point in $\Delta(\Gamma)/\Gamma$ is the center of an open non-Euclidean disc in U/Γ of radius $\varepsilon_1/2$. The area of such a disc equals $4\pi\sinh^2(\varepsilon_1/2)$.

Let $d = \mathrm{diam}^{\#}\Gamma$ and let Q, Q' be two points on $\Delta(\Gamma)/\Gamma$ whose distance in $\Delta(\Gamma)/\Gamma$ is d. We join Q to Q' by the unique shortest geodesic C in U/Γ, and modify C to obtain a path C_0 in $\Delta(\Gamma)/\Gamma$, leading from Q to Q' and consisting of at most $n+1$ arcs $A_1, \cdots, A_r$ of C and at most n boundary arcs of the regions D_j. Let A_1 be the arc of maximum length among the A_j, and denote this length by d_1. Then the length of C_0 is at most $(n+1)d_1 + n$, so that

$$d \leqq (n+1)d_1 + n. \tag{13}$$

Since C is the shortest path between any of its two points, there are at least

$$N = [d_1/\varepsilon_1] + 1 \geqq d_1/\varepsilon_1 \tag{14}$$

points on A_1 such that the open discs of radius $\varepsilon_1/2$ about these points do not intersect. Hence

$$N 4\pi \sinh^2(\varepsilon_1/2) \leqq A(\sigma). \tag{15}$$

Inequalities (13), (14), (15) show that

$$d \leqq \frac{(n+1)A(\sigma)}{4\pi} \frac{\varepsilon_1}{\sinh^2(\varepsilon_1/2)} + n, \tag{16}$$

which, together with (11), implies (10).

Assume next that Γ has elements of finite order. By the Fenchel-Fox theorem (see [3]), Γ has a torsion free subgroup $\hat{\Gamma}$ of finite index. This $\hat{\Gamma}$ is a Fuchsian group of signature $\hat{\sigma}, \hat{\sigma}$ depending only on σ. The Poincaré metric in U induces metrics ds on U/Γ and $d\hat{s}$ or $U/\hat{\Gamma}$ and the natural projection $U/\hat{\Gamma} \to U/\Gamma$ takes $d\hat{s}$ into ds. This projection also maps $\Delta(\hat{\Gamma})/\hat{\Gamma}$ onto $\Delta(\Gamma)/\Gamma$, so that $\mathrm{diam}^{\#}\Gamma \leqq \mathrm{diam}^{\#}\hat{\Gamma}$. Since Lemma 5 holds for $\hat{\Gamma}$, it also holds for Γ.

Proof of Theorem 1′. Note that the subset considered is closed and use Lemmas 5 and 4.

We remark that Lemma 5 has a partial converse.

Lemma 6. *Let Γ be a Fuchsian group of signature σ and assume that $\mathrm{diam}^{\#}\Gamma \leqq \alpha$. Then $|\mathrm{trace}\,\gamma| \geqq 2 + \varepsilon > 2$ for all hyperbolic $\gamma \in \Gamma$, where ε depends only on α and σ.*

This can be proved from Lemma 4 by verifying that the infimum of $|\operatorname{trace}\gamma|$, γ in Γ and hyperbolic, is a continuous function on $X(\sigma)$. It would be desirable, however, to have a direct proof and an explicit estimate for ε.

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